Complex Number

Roots of unity and Factorization

1. If $1, \omega, \omega^2$ are the cube roots of unity, prove that :

(i) $(a + \omega - \omega^2)(a - \omega + \omega^2) = a^2 + 3$,

(ii) $(1 + i\omega - \omega^2)(1 - \omega + i\omega^2) = 2$,

(iii) $(a + b)(a + b\omega)(a + b\omega^2) = a^3 + b^3$,

(iv) $(a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) = a^3 + b^3 + c^3 - 3abc$,

2. Let ω be a complex cube root of unity and n is a positive integer, but not a multiple of 3 .

(i) Prove that $1 + \omega^n + \omega^{2n} = 0$.

(ii) Prove that $x^2 + y^2 + z^2 - xy - yz - zx$ has a linear factor $x + \omega y + \omega^2 z$.

(iii) Deduce that $x^2 + y^2 + z^2 - xy - yz - zx$ is a factor of $(x - y)^n + (y - z)^n + (z - x)^n$.

3. If ω is one of the complex cube roots of unity, show that

$$(x + a + b)(x + \omega a + \omega^2 b)(x + \omega^2 a + \omega b) \equiv x^3 - 3abx + a^3 + b^3 .$$

Hence solve the equation $x^3 - px + q = 0$ by putting $p = 3ab$ and $q = a^3 + b^3$.

4. If $(1 + x)^n = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$, n being a positive integer,

(a) By setting $x = 1, \omega, \omega^2$ in turn and adding show that , $c_0 + c_3 + c_6 + \dots = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$

(b) Show that : $(c_0 + c_3 + \dots)^2 + (c_1 + c_4 + \dots)^2 + (c_2 + c_5 + \dots)^2 = \frac{1}{3}\left(4^n + 2\right)$.

5. Find the quadratic factors with real coefficients of

(i) $x^8 - 4x^4 + 16$

(ii) $x^6 + 8x^3 + 64$

6. Factorize $x^9 - 1$ and prove that

(i) $x^3 + \frac{1}{x^3} + 1 = \left(x + \frac{1}{x} - 2\cos\frac{2\pi}{9}\right)\left(x + \frac{1}{x} - 2\cos\frac{4\pi}{9}\right)\left(x + \frac{1}{x} - 2\cos\frac{8\pi}{9}\right)$,

(ii) $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{4\pi}{9} = \frac{\sqrt{3}}{8}$,

(iii) $64\left(\cos^2\theta - \cos^2\frac{\pi}{9}\right)\left(\cos^2\theta - \cos^2\frac{2\pi}{9}\right)\left(\cos^2\theta - \cos^2\frac{4\pi}{9}\right) = 4\cos^2 3\theta - 1$.

7. **(i)** Show that $x^{2n} - 1 = (x - 1)(x + 1)\prod_{k=1}^{n-1}\left[x - \operatorname{cis}\frac{k\pi}{n}\right]\left[x - \operatorname{cis}\left(-\frac{k\pi}{n}\right)\right]$,

(ii) Express $\frac{1}{x^{2n} - 1}$ in partial fractions with **real** linear and quadratic denominators.

8. Solve the equation $(z+1)^8 - z^8 = 0$, and prove that $(z+1)^8 - z^8 = \frac{1}{16}(2z+1)\prod_{k=1}^{3}\left\{4z^2+4z+\csc^2\frac{k\pi}{8}\right\}$.

Hence show that $16\left(\cos^{16}\theta - \sin^{16}\theta\right) = \cos 2\theta \prod_{k=1}^{3}\left\{\cos^2 2\theta + \cot^2\frac{k\pi}{8}\right\}$.

9. Show that all the **non-zero** roots of the equation $(1+x)^{2n+1} = (1-x)^{2n+1}$ are given by

$\pm i\tan\frac{r\pi}{2n+1}$, where r has values $1, 2, \ldots, n$.

By putting $n = 2$ or otherwise, show that $\tan^2\frac{\pi}{5}\times\tan^2\frac{2\pi}{5} = 5$.

10. Show that every root of the equation $(z+1)^{2n} + (z-1)^{2n} = 0$, where n is a positive integer, is purely imaginary.

If the roots are represented in the Argand diagram by points $P_1, P_2, \ldots, P_{2n}$, prove that, if O is the origin, $OP_1^2 + OP_2^2 + \ldots + OP_{2n}^2 = 2n(2n-1)$.

11. If n is a positive integer, prove that

(i) $x^{2n} - 1 = (x-1)(x+1)\prod_{r=1}^{n-1}\left(x^2 - 2x\cos\frac{r\pi}{n} + 1\right)$

(ii) $x^{2n+1} - 1 = (x-1)\prod_{r=1}^{n}\left(x^2 - 2x\cos\frac{2r\pi}{2n+1} + 1\right)$

Deduce from **(i)** that, if $x \neq 0$, $x^n - x^{-n} = \left(x - x^{-1}\right)\prod_{r=1}^{n-1}\left(x + x^{-1} - 2\cos\frac{r\pi}{n}\right)$.

Use the last result to prove that $\prod_{r=1}^{n-1}\sin\frac{r\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}$.

12. Prove that $x^{2n} - 2x^n\cos n\theta + 1 = \prod_{k=0}^{n-1}\left[x^2 - 2x\cos\left(\theta + \frac{2k\pi}{n}\right) + 1\right]$.

Deduce the following results :

(i) $\sin n\alpha = 2^{n-1}\prod_{k=0}^{n-1}\sin\left(\alpha + \frac{k\pi}{n}\right)$

(ii) $\cos n\alpha - \cos n\beta = 2^{n-1}\prod_{k=0}^{n-1}\left[\cos\alpha - \cos\left(\beta + \frac{2k\pi}{n}\right)\right]$.

(iii) From **(i),** deduce, by logarithmic differentiation,

$$\cot n\alpha = \frac{1}{n}\sum_{k=0}^{n-1}\cot\left(\alpha + \frac{k\pi}{n}\right), \qquad \alpha \neq \frac{r\pi}{n},$$

(iv) Deduce from **(iii)** that $\csc^2 n\theta = \frac{1}{n^2}\sum_{k=0}^{n-1}\csc^2\left(\theta + \frac{k\pi}{n}\right), \qquad \theta \neq \frac{r\pi}{n}$.

13. **(i)** If n is a positive integer, prove that

$$x^{2n} - 2x^n a^n \cos n\theta + a^{2n} = \prod_{k=0}^{n-1}\left[x^2 - 2xa\cos\left(\theta + \frac{2k\pi}{n}\right) + a^2\right] \quad .$$

(ii) By taking $x = a = 1$ and in turn, $\theta = 2\alpha$ and $\theta = 2\alpha - \pi$, show that the value of

$$\prod_{r=0}^{n-1}\sin^2\left(\alpha + \frac{r\pi}{n}\right) + \prod_{r=0}^{n-1}\cos^2\left(\alpha + \frac{r\pi}{n}\right)$$

is 2^{2-2n} when n is odd, and $2^{3-2n}\sin^2 n\alpha$ when n is even.

(iii) Using derivatives, deduce from (i) that

$$\frac{nx^{n-1}\left(x^n - a^n\cos n\theta\right)}{x^{2n} - 2x^n a^n \cos n\theta + a^{2n}} = \sum_{r=0}^{n-1}\frac{x - a\cos\left(\theta + \frac{2r\pi}{n}\right)}{x^2 - 2xa\cos\left(\theta + \frac{2r\pi}{n}\right) + a^2} \quad .$$

(iv) By taking the derivatives again and substituting a particular value for x , show that, when $\cos n\theta \neq 1$,

$$\sum_{r=0}^{n-1}\frac{1}{1 - \cos\left(\theta + \frac{2r\pi}{n}\right)} = \frac{n^2}{1 - \cos n\theta} \quad .$$

(v) In **(i)** , take $x = i$, $a = 1$ and n even, deduce that, if k is a positive integer,

$$2^{2k-1}\cos\theta\cos\left(\theta + \frac{\pi}{k}\right)\cos\left(\theta + \frac{2\pi}{k}\right) \ldots \cos\left(\theta + \frac{(2k-1)\pi}{k}\right) = (-1)^k - \cos 2k\theta \quad .$$

and deduce that

$$2^{2k-1}\sin\theta\sin\left(\theta + \frac{\pi}{k}\right)\sin\left(\theta + \frac{2\pi}{k}\right) \ldots \sin\left(\theta + \frac{(2k-1)\pi}{k}\right) = (-1)^k\left(1 - \cos 2k\theta\right) \quad .$$

(vi) Prove that :

$$\prod_{r=0}^{n-1}\sin\frac{(4r+1)\pi}{4n} = 2^{\frac{1}{2}-n} \quad .$$

14. The n points $A_0 , A_1 , A_2 , \ldots , A_{n-1}$ are the vertices of a regular polygon of n sides which is inscribed in a circle center O , radius a . P is a point such that $OP = x$ and the angle $POA_0 = \theta$.

Prove that $\prod_{k=0}^{n-1} PA_r = \sqrt{x^{2n} - 2x^n a^n \cos n\theta + a^{2n}}$